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ON A GENERALIZATION OF THE PLANK PROBLEM

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A *strip* or a *plank* S in \mathbb{R}^n is a closed set bounded by two parallel hyperplanes. The distance of these hyperplanes is called the *width* of S . The *minimal width* of a convex closed set K is the minimal width of a strip containing K .

The following theorem was conjectured by A. Tarski in 1932 and proved by T. Bang [2] in 1951:

If a closed convex set K in \mathbb{R}^n is covered by a finite number of strips, then the sum of their widths is greater than or equal to the minimal width of K .

This result has recently been generalized to Banach spaces by K. Ball [1].

If $n = 2$ and K is the unit disc then there is an extremally simple proof for the above result.

Assume that the unit disc in \mathbb{R}^2 is covered by strips S_1, \dots, S_k with widths d_1, \dots, d_k . Without loss of generality we can also assume that both bounding lines of the strips intersect the unit circle. Now we consider the unit sphere in \mathbb{R}^3 and to each strip S_i in \mathbb{R}^2 we construct a three dimensional strip S_i^* which is of width d_i and intersects the xy -plane in S_i . Since S_1, \dots, S_k cover the unit disc, hence S_1^*, \dots, S_k^* cover the unit sphere. The area of the intersection of the unit sphere and the strip S_i^* is $2\pi d_i$ independently of the position of the i -th strip. (This is a well known fact from calculus, already discovered by Archimedes.) Thus the sum of these areas exceeds the area of the unit sphere, i.e.

$$\sum_{i=1}^k 2\pi d_i \geq 4\pi \quad \implies \quad \sum_{i=1}^k d_i \geq 2,$$

which was to be proved.

We can interpret this proof in the following way: If S is a subset of the disc then we project it up to the sphere, measure the area of the projection and call this number the μ measure of S . Then the μ measure of a the intersection of a strip and the disc is the width of the strip times 2π . Then the statement is a simple consequence of the subadditivity of μ .

In what follows, we generalize this idea and extend the result discussed above.

An *angular domain* in \mathbb{R}^2 is a closed convex set D bounded by two halflines. The *angle* of D is the angle closed by the bounding halflines. The *vertex* of D is the common endpoint of these two halflines.

Theorem 1. *Let two concentric circles k and K be given on the plane with radii r and R , $r < R$. Assume that the disc bounded by k is covered by angular domains whose vertices are within K . Then the sum of the angles of these angular domains is greater than or equal to the view angle of k from an arbitrary point of K .*

Remark. This result was proposed as a problem by the author on the 1985 M. Schweitzer competition (see [3]).

Proof. Denote by O the common center of the circles and by D_1, \dots, D_k the given angular domains with angles $\alpha_1, \dots, \alpha_k$. An angular domain D will be called *regular* if the vertex of D is on K and both bounding halflines of D intersect k . Without loss of generality, we can assume that D_1, \dots, D_k are regular domains.

The idea of the proof is the following: We construct a rotation invariant nonnegative measure μ on the closed disc T bounded by k such that the measure of the intersection of D and T is α , where D is an arbitrary regular angular domain with angle α . Having such a measure we can give a one line proof for the theorem:

$$\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \mu(D_i \cap T) \geq \mu\left(\bigcup_{i=1}^k (D_i \cap T)\right) = \mu(T),$$

and observe that $\mu(T)$ is exactly the view angle of k from any point of K .

Now we construct the desired measure. Let $P \in T$ be an arbitrary point, and denote by ρ the distance of P and O . Then define

$$F(P) = f(\rho) = \frac{1}{\pi} \cdot \frac{1}{R^2 - \rho^2} \cdot \sqrt{\frac{R^2 - r^2}{r^2 - \rho^2}}.$$

If S is a Lebesgue measurable subset of T then let

$$\mu(S) = \int_S F(P) dP.$$

Obviously μ is a rotation invariant nonnegative measure on T . To prove the key property of μ , let D be a regular angular domain with vertex A and angle α . Then we want to show $\mu(D) = \alpha$. Without loss of generality we can assume that one of the bounding halflines of D is tangent to k

at the point Q . (In the general case D can be obtained as the difference of two such angular domains.) Denote by ε the angle $OAQ < \pi$ and by d the (signed) distance of the other bounding halfline from O . (This distance is positive if O is outside D , and negative if O is inside D .) Then $d = R \sin(\varepsilon - \alpha)$. Using successive integration, we obtain

$$\begin{aligned} \mu(D) &= \int_D F(P) dP = \int_d^r \int_{-\arccos(d/\rho)}^{\arccos(d/\rho)} f(\rho) \rho d\varphi d\rho \\ &= \int_d^r 2f(\rho) \rho \arccos(d/\rho) d\rho \\ &= \int_{R \sin(\varepsilon - \alpha)}^r 2f(\rho) \rho \arccos\left(\frac{R \sin(\varepsilon - \alpha)}{\rho}\right) d\rho. \end{aligned}$$

Thus we have to show that

$$\int_{R \sin(\varepsilon - \alpha)}^r 2f(\rho) \rho \arccos\left(\frac{R \sin(\varepsilon - \alpha)}{\rho}\right) d\rho = \alpha$$

for all $0 \leq \alpha \leq 2\varepsilon$. Substituting the new variable $t = R \sin(\varepsilon - \alpha)$ this reduces to

$$\int_t^r 2f(\rho) \rho \arccos(t/\rho) d\rho = \varepsilon - \arcsin(t/R),$$

for $-r \leq t \leq r$. This latter equation is obviously valid for $t = r$, thus it suffices to show that the derivatives of both sides with respect to t are identical, i.e.

$$\int_t^r \frac{2f(\rho) \rho}{\sqrt{\rho^2 - t^2}} d\rho = \frac{1}{\sqrt{R^2 - t^2}}, \quad -r < t < r.$$

However

$$\begin{aligned} \int_t^r \frac{2f(\rho) \rho}{\sqrt{\rho^2 - t^2}} d\rho &= \int_t^r \frac{2}{\pi} \cdot \frac{\rho}{R^2 - \rho^2} \cdot \sqrt{\frac{R^2 - r^2}{(r^2 - \rho^2)(\rho^2 - t^2)}} d\rho \\ &= \left[\frac{2}{\pi} \cdot \frac{1}{\sqrt{R^2 - t^2}} \cdot \arctan \sqrt{\frac{R^2 - r^2}{R^2 - t^2} \cdot \frac{\rho^2 - t^2}{r^2 - \rho^2}} \right]_{\rho=t}^{\rho=r} \\ &= \frac{1}{\sqrt{R^2 - t^2}}. \end{aligned}$$

Thus the proof is complete.

Remark. When $n = 2$ and K is the unit disc, then the statement of the plank problem can easily be derived from our theorem. Denote the unit disc by T and assume that it is covered by strips S_1, \dots, S_k (whose bounding lines intersect T). Take a concentric circle K with radius R , where R is sufficiently large. Assume that the two bounding lines of S_i intersect K in A_i, B_i and in C_i, D_i . We choose the notation such that S_i is covered by the two regular angular domains $A_i B_i D_i <$ and $B_i D_i C_i <$. Denote by α'_i and α''_i their angle and by d_i the width of S_i . Then we have

$$\frac{d_i}{2R-2} \geq \tan \alpha'_i \geq \alpha'_i, \quad \frac{d_i}{2R-2} \geq \tan \alpha''_i \geq \alpha''_i.$$

Thus the theorem yields

$$2 \sum_{i=1}^k \frac{d_i}{2R-2} \geq \sum_{i=1}^k (\alpha'_i + \alpha''_i) \geq 2\varepsilon \geq 2 \sin \varepsilon \geq \frac{2}{R}.$$

Now taking the limit $R \rightarrow \infty$ we obtain the statement.

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